A slender-body theory for ship oscillations in waves

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A linearized theory is developed for the oscillations of a slender body which is floating on the free surface of an ideal fluid, in the presence of incident plane progressive waves. Green's theorem is used to represent the velocity potential and the first-order slender-body potential is developed from asymptotic approximation. The general theory is valid for arbitrary slender bodies in oblique waves, and detailed results are presented for a body of revolution.

1. Introduction

Efforts to analyse the hydrodynamical characteristics of an oscillating ship in waves have been divided primarily between the two approaches of two-dimensional 'strip theory' and three-dimensional 'thin-ship' theory. The strip theory is semi-empirical in nature and moreover does not provide insight into forward speed effects or longitudinal interference effects, while the thin ship theory leads to relatively trivial first-order effects and highly complex second-order effects whose complete validity has not been verified. Detailed discussions of this situation can be found in the papers of Newman (1961), Ursell (1962), and Vossers (1962).

It has been apparent for some years that a slender-body theory similar to that developed in aerodynamics might provide the remedy to this unsatisfactory situation. Recently there have been several papers with this goal, directed towards both the oscillatory problem in waves and the steady-state problem of wave resistance in calm water. We shall restrict our attention here to the simplest unsteady problem of an oscillatory ship or floating slender body having no forward velocity. This is precisely the problem treated by Ursell (1962), for the special case of a body of revolution performing pitch and heave oscillations. Our aim in the present paper is to obtain a more general theory valid for non-axisymmetric bodies, and allowing for the presence of an oblique incident wave system with the resulting transverse as well as longitudinal oscillations. It should be noted that the present theory is related to that of Grim (1957, 1960), which is based primarily on physical concepts.

Our analysis is based upon the slender-body approach suggested by Vossers (1962), wherein Green's theorem is employed to formulate the exact problem, and a consistent asymptotic approximation is obtained retaining terms of first order in the slenderness parameter. This method is quite general since it permits the study of higher-order effects and end effects, and it can be applied to the problem with forward speed as well. A further advantage of Green's theorem in the present study is to demonstrate the relation between the free-surface problem and its zero frequency limit, corresponding to a double body† in an infinite fluid.

It will be seen, moreover, that while a consistent first-order velocity potential can be obtained, the resulting equations of motion for the oscillations of the body in waves are comparatively trivial; the dominant longitudinal forces consist only of hydrostatic restoring forces plus the exciting forces due to the undisturbed incident wave system, while for transverse motions the simple strip theory is recovered.

2. Basic formulation of the problem

We wish to consider the oscillatory fluid motion due to a slender body which is floating on the free surface of an inviscid, incompressible fluid, in the presence of incident plane progressive waves. It is assumed that all oscillatory motions of the body and the fluid are small and the problem can be solved within the framework of linearized water-wave theory. Let (x, y, z) be a rectangular co-ordinate system with $z = 0$ the plane of the undisturbed free surface and z positive upwards. The body is taken to be of length L and maximum transverse dimension ϵL, where ϵ is a small parameter representing the slenderness of the body. The body axis coincides with the segment $(-\frac{1}{2}L \leqslant x \leqslant \frac{1}{2}L)$ of the x-axis, and as $\epsilon \to 0$ the body tends in the limit to this segment.

Assuming linearized simple harmonic motion the velocity potential, whose positive gradient represents the fluid velocity vector, may be written in the form

$$\Phi(x,y,z,t) = \phi(x,y,z)\,e^{-i\omega t}$$
$$= [\phi_i(x,y,z)+\phi_b(x,y,z)]e^{-i\omega t}, \tag{2.1}$$

where the real part is understood in all expressions involving $e^{-i\omega t}$. The potential ϕ_i denotes the known incident wave potential and ϕ_b the unknown disturbance due to the presence of the body. The latter consists of both the disturbance due to the body's oscillations (in calm water) and the diffraction of the incident wave system by the (fixed) body.

If the incident wave amplitude is A and the wave system is moving in a direction of angle β relative to the x-axis, the incident wave potential is

$$\phi_i = \frac{gA}{\omega}\exp\left[K(z+ix\cos\beta+iy\sin\beta)\right], \tag{2.2}$$

where $K = \omega^2/g$ is the wave-number. It is assumed throughout that $KL = O(1)$ with respect to ϵ as $\epsilon \to 0$. The potential ϕ_b is a solution of Laplace's equation satisfying the linearized free-surface condition

$$K\phi_b - \frac{\partial \phi_b}{\partial z} = 0 \quad \text{on} \quad z = 0, \tag{2.3}$$

† Throughout the paper the 'double-body problem' implies a rigid free-surface condition $\partial\phi/\partial z = 0$, or by reflexion the problem of a double body in an infinite fluid. However, the latter must have as its normal velocity an even function of z, and thus for vertical motions (pitch and heave) a pulsation of the double body is implied.

and the kinematic boundary condition

$$\frac{\partial \phi_b}{\partial n} = V_n - \frac{\partial \phi_i}{\partial n} \quad \text{on} \quad S, \tag{2.4}$$

where S is the submerged surface of the body in its mean position and V_n is its normal velocity. In addition the potential ϕ_b must satisfy a radiation condition of outgoing waves at infinity and its gradient must vanish as $z \to -\infty$.

From Green's theorem the potential ϕ_b at any point in the fluid may be represented in the form

$$\phi_b(x,y,z) = \frac{1}{4\pi}\iint_S \left[G(x,y,z;\xi,\eta,\zeta)\frac{\partial}{\partial n}\phi_b(\xi,\eta,\zeta) - \phi_b(\xi,\eta,\zeta)\frac{\partial}{\partial n}G(x,y,z;\xi,\eta,\zeta)\right] dS, \tag{2.5}$$

where the integral is over the surface S, the direction of the normal n is into the body, and the Green's function is defined (cf. Wehausen & Laitone 1961) by the expression

$$G = G_0 + G_1, \tag{2.6}$$

$$G_0 = [(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]^{-\frac{1}{2}} + [(x-\xi)^2+(y-\eta)^2+(z+\zeta)^2]^{-\frac{1}{2}}, \tag{2.7}$$

$$G_1 = 2K \int_0^\infty \frac{dk}{k-K}\, e^{k(z+\zeta)} J_0(k[(x-\xi)^2+(y-\eta)^2]^{\frac{1}{2}}). \tag{2.8}$$

The contour of integration in the integral for G_1 is indented below the singularity $k = K$, in order to satisfy the radiation condition of outgoing waves at infinity. Physically the Green's function G represents the potential of an oscillatory source, located beneath the free surface at the point $x = \xi$, $y = \eta$, $z = \zeta$; the function G_0 is the elementary source function $1/R$ plus its image above the free surface, and the function G_1 represents the necessary correction to account for free-surface effects.

3. Results of the slender-body approximation

A slender-body approximation for the velocity potential will be derived in § 4, but before proceeding to the derivation it may be advantageous to first present the more important results. It will be shown that if the field point (x, y, z) is in the 'near field', i.e. a distance $O(\epsilon)$ from the body surface, then (2.5) has the limit

$$\phi_b \simeq \frac{1}{4\pi}\iint_S \left(G_0 \frac{\partial \phi_b}{\partial n} - \phi_b \frac{\partial G_0}{\partial n}\right) dS + F(x), \tag{3.1}$$

where

$$F(x) = \lim_{\epsilon\to 0}\frac{1}{4\pi}\iint_S G_1 \frac{\partial \phi_b}{\partial n} dS$$
$$= \frac{K}{4}\int_{-\frac{1}{2}L}^{\frac{1}{2}L}\left(\int_C \frac{\partial \phi_b}{\partial n} dl\right)\{-\mathbf{H}_0(K|x-\xi|) - Y_0(K|x-\xi|) + 2iJ_0(K|x-\xi|)\} d\xi. \tag{3.2}$$

In these equations the error is a factor $1+O(\epsilon\log\epsilon)$. The integral

$$\int_C \frac{\partial\phi_b}{\partial n}\,dl$$

in (3.2) is over the submerged contour of the body in the transverse plane $\xi = \text{const.}$, and it corresponds physically to the fluid flux through this contour. The functions J_0 and Y_0 are the Bessel functions of the first and second kind, respectively, and $\mathbf{H}_0$ is the Struve function.

Equation (3.1), which clearly is related to the double-body problem, can be reduced further by considering the limit of surface integrals with the kernels G_0 and $\partial G_0/\partial n$. Thus, to the same degree of approximation, (3.1) is equivalent to

$$\phi_b \simeq -\frac{1}{2\pi}\int_{C+\bar{C}}\left(\frac{\partial\phi_b}{\partial n}-\phi_b\frac{\partial}{\partial n}\right)\log\frac{\rho}{L}\,dl+f(x), \tag{3.3}$$

where
$$f(x) = F(x)+\frac{1}{2\pi}\int_{-\frac{1}{2}L}^{\frac{1}{2}L}\log\frac{2|x-\xi|}{L}\operatorname{sgn}(x-\xi)\frac{\partial}{\partial\xi}\left(\int_C\frac{\partial\phi_b}{\partial n}\,dl\right)d\xi, \tag{3.4}$$

and $\rho^2 = (y-\eta)^2+(z-\zeta)^2$. The contour $\bar{C}$ is the image of C above the plane $z = 0$, and $\operatorname{sgn}(x-\xi) = \pm 1$, according as $x-\xi \gtrless 0$, respectively.

Finally, the solution of these equations will be established in the two equivalent forms

$$\phi_b(x,y,z) \simeq \phi_0(x,y,z)+F(x) \tag{3.5}$$

and

$$\phi_b(x,y,z) \simeq \phi_{2D}(y,z;x)+f(x), \tag{3.6}$$

provided the field point is in the near field or on the body surface itself. In (3.5) the potential ϕ_0 is the three-dimensional solution of the double-body problem, satisfying the boundary condition (2.4) on S and the rigid free-surface condition $\partial\phi/\partial z = 0$ on $z = 0$. Similarly, in (3.6) the potential ϕ_{2D} is the two-dimensional solution of the same double-body problem in the transverse plane $x = \text{const.}$†

The two results (3.5) and (3.6) have been presented separately since (3.5) shows more clearly the role of the free surface effects, or the relation between the potentials ϕ_b and ϕ_0, while (3.6) is the logical result of the complete slender-body approximation. Thus the function $F(x)$, defined by equation (3.2), represents the effects of the free surface (and the dependence on the wave-number K) while the function $f(x)$, defined by (3.4), contains in addition the usual interaction integral resulting from the slender-body approximation of the potential ϕ_0.

It should be emphasized that the potentials given by (3.5) and (3.6) can hold only in the near field (and on the body). This is obvious from the fact that they satisfy neither the free-surface condition nor Laplace's equation, except in the limiting sense for small values of the co-ordinates y and z. An expression for the potential valid throughout the fluid domain can be obtained by substituting (3.5) or (3.6) in Green's theorem (2.5), but this procedure is unnecessary if one is concerned only with the near field, as in considering the pressure distribution at the body.

† Actually it is necessary to be more specific since the potential ϕ_{2D} is arbitrary, to the extent of adding a non-trivial constant. It is sufficient to require that

$$\phi_{2D} \sim C\log(y^2+z^2)/L^2+o(1), \quad \text{for} \quad y^2+z^2 \to \infty,$$

with a suitably chosen constant C.

We note that if $\partial\phi_b/\partial n$ is an odd function of y, as in transverse oscillations, then, from (3.2), the function $F(x)$ vanishes. Thus to leading order the near field potential for transverse or asymmetric modes is identical to the corresponding potential for the double body in an infinite fluid. On the other hand for symmetric disturbances, as in pitch and heave, the function $F(x)$ will be non-zero and the effects of the free surface will appear in the first-order potential. This result has been deduced from physical reasoning by Grim (1957).

4. Derivation of the slender-body potential

In order to derive the results presented in § 3 we shall require the near field potentials of simple source and normal dipole distributions on the submerged surface of a slender body. These relations are fundamental to the derivation of slender-body theories from Green's theorem and are derived in the Appendix. The results are as follows:†

$$\iint_S \sigma(S)\,[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]^{-\frac{1}{2}}\,dS$$
$$\simeq \int_{-\frac{1}{2}L}^{\frac{1}{2}L} \log\frac{2|x-\xi|}{L}\,\operatorname{sgn}(x-\xi)\,\frac{\partial}{\partial\xi}\left(\int_C \sigma(\xi,l)\,dl\right)d\xi$$
$$-2\int_C \sigma(x,l)\log(\rho/L)\,dl, \qquad (4.1)$$

and
$$\iint \mu(S)\,\frac{\partial}{\partial n}\,[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]^{-\frac{1}{2}}\,dS$$
$$\simeq -2\int_C \mu(x,l)\,\frac{\partial}{\partial n}\log(\rho/L)\,dl. \qquad (4.2)$$

These approximations are valid both for an open surface S, such as for a floating body where S is bounded by the plane $z=0$, in which case the contour C is open, and also for a closed surface S such as a submerged body or one in an infinite fluid, in which case the contour C is closed. The error in (4.1) and (4.2) is connected with 'end effects', depending both on the proximity of the field point to the ends of the body and on the geometrical nature of the body ends, i.e. whether they are blunt, pointed or cusped. For the present application it is sufficient to say that the error is a factor $1+O(\epsilon)$ everywhere for pointed or cusped bodies, and except near the ends for blunt bodies. Further details can be inferred from the Appendix.

It is now a straightforward matter to derive the results of § 3. Let us first consider the limiting form of the function G_1, defined by equation (2.8), and its normal derivative. Since $(z+\zeta)$ and $(y-\eta)$ are both $O(\epsilon)$, the asymptotic limit of G_1 is

$$G_1 \simeq 2K \int_0^\infty J_0(k|x-\xi|)\,\frac{dk}{k-K}$$
$$= \pi K\{-\mathbf{H}_0(K|x-\xi|)-Y_0(K|x-\xi|)+2iJ_0(K|x-\xi|)\}, \qquad (4.3)$$

† In equation (4.2) the length parameter L is arbitrary, and it is inserted only because ρ has the dimensions of length. In fact, the same is true in (4.1), provided the same length parameter is used in both terms on the right-hand side.

with an error of order $(\epsilon\nabla G_1)$. The notation is that of Watson (1952), and the integrals involved can be evaluated from the results of §§ 13.2 and 13.6 therein.

Similarly, assuming differentiation under the integral sign is permissible,

$$\begin{aligned}\frac{\partial G_1}{\partial \zeta} &\cong 2K\int_0^\infty e^{k(z+\zeta)} J_0(k[(x-\xi)^2+(y-\eta)^2]^{\frac{1}{2}})\,dk \\ &\quad + 2K^2 \int_0^\infty J_0(k|x-\xi|)(k-K)^{-1}dk \\ &= 2K[(x-\xi)^2+(y-\eta)^2+(z+\zeta)^2]^{-\frac{1}{2}} \\ &\quad + \pi K^2\{-\mathbf{H}_0(K|x-\xi|)-Y_0(K|x-\xi|)+2iJ_0(K|x-\xi|)\}, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\frac{\partial G_1}{\partial \eta} &\cong \frac{2K(y-\eta)}{[(x-\xi)^2+(y-\eta)^2]^{\frac{1}{2}}}\int_0^\infty e^{k(z+\zeta)} J_1(k[(x-\xi)^2+(y-\eta)^2])\,dk \\ &\quad + \frac{2K^2(y-\eta)}{[(x-\xi)^2+(y-\eta)^2]^{\frac{1}{2}}}\int_0^\infty J_1(k|x-\xi|)\frac{dk}{k-K} \\ &= \frac{2K(y-\eta)}{[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]^{\frac{1}{2}}\{[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]^{\frac{1}{2}}+(z+\zeta)\}} \\ &\quad + \frac{\pi K^2(y-\eta)}{[(x-\xi)^2+(y-\eta)^2]^{\frac{1}{2}}}\Big\{\mathbf{H}_{-1}(K|x-\xi|)-Y_1(K|x-\xi|) \\ &\qquad -\frac{2}{\pi K|x-\xi|}+2iJ_1(K|x-\xi|)\Big\}. \end{aligned} \tag{4.5}$$

The functions represented by the sums within braces in equations (4.3), (4.4) and (4.5) are integrable, having at most logarithmic singularities.

From the limiting form (4.3) for G_1 it follows directly that

$$\begin{aligned}\iint_S G_1 \frac{\partial \phi_b}{\partial n}\,dS &\cong \pi K\int_{-\frac{1}{2}L}^{\frac{1}{2}L}\left(\int_C \frac{\partial \phi_b}{\partial n}\,dl\right)\{-\mathbf{H}_0(K|x-\xi|) \\ &\quad -Y_0(K|x-\xi|)+2iJ_0(K|x-\xi|)\}d\xi \\ &\equiv 4\pi F(x). \end{aligned} \tag{4.6}$$

Since $\int_C dl = O(\epsilon)$ and $\partial\phi_b/\partial n = V_n - \partial\phi_i/\partial n = O(1)$,

it follows that $F(x) = O(\epsilon)$. The error in (4.6) is

$$O\left(\epsilon\iint_S \frac{\partial\phi_b}{\partial n}\nabla G_1\,dS\right) = O(\epsilon^2\log\epsilon),$$

where the second estimate follows from an analysis similar to that given below in the next paragraph. Thus the error in (4.6) is a factor $1+O(\epsilon\log\epsilon)$.

Next we must dispose of the integral

$$\iint_S \phi_b\frac{\partial G_1}{\partial n}\,dS = \iint_S \phi_b\left[\frac{\partial G_1}{\partial \zeta}\cos(n,\zeta)+\frac{\partial G_1}{\partial \eta}\cos(n,\eta)\right]dS,$$

by showing that it is $O(\phi_b\,\epsilon\log\epsilon)$. Since the area of the surface S is $O(\epsilon)$, the proof is trivial for all terms in (4.4) and (4.5) which are bounded as $\epsilon\to 0$, or as in the case of the second summand of (4.5), products of an integrable function of $(x-\xi)$ times a function which is bounded as $\epsilon \to 0$. Thus we need consider only

$$\iint_S \phi_b \frac{\partial G_1}{\partial n} dS \cong 2K \iint_S \left[\cos(n,\zeta) + \frac{(y-\eta)\cos(n,\eta)}{[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]^{\frac{1}{2}}+z+\zeta}\right] \times \frac{\phi_b(\xi,\eta,\zeta)}{[(x-\xi)^2+(y-\eta)^2+(z+\zeta)^2]^{\frac{1}{2}}} dS.$$

This integral can be regarded as a surface source distribution, of density

$$\sigma = \left[\cos(n,\zeta) + \frac{(y-\eta)\cos(n,\eta)}{[(x-\xi)^2+(y-\eta)^2+(z+\zeta)^2]^{\frac{1}{2}}+z+\zeta}\right]\phi_b(\xi,\eta,\zeta).$$

Since the quantity in square brackets is continuous and bounded, as $\epsilon \to 0$, it follows from (4.1) that

$$\iint \phi_b \frac{\partial G_1}{\partial n} dS = O(\phi_b \epsilon \log \epsilon).$$

Substituting this result and equation (4.6) in the exact form of Green's theorem, (2.5), we obtain the asymptotic result

$$\phi_b \cong \frac{1}{4\pi}\iint_S \left(G_0 \frac{\partial \phi_b}{\partial n} - \phi_b \frac{\partial G_0}{\partial n}\right) dS + F(x), \tag{4.7}$$

which is valid for field points in the near field, with the error a factor $1+O(\epsilon \log \epsilon)$

Equation (4.7) will be recognized as the integral equation† (i.e. Green's theorem) corresponding to the solution ϕ_0 of the double-body problem, modified by the free term $F(x)$. Thus the solution $\phi_b = \phi_0 + F(x)$ can be anticipated, but to verify this conclusion we shall reduce the surface integral appearing in (4.7) to its slender-body limit, using the equations (4.1) and (4.2) for source and dipole distributions. The result, by direct substitution of (4.1) and (4.2) in (4.7), is

$$\phi_b \cong -\frac{1}{2\pi}\int_{C+\bar{C}} \left(\frac{\partial \phi_b}{\partial n} - \phi_b \frac{\partial}{\partial n}\right) \log(\rho/L)\, dl + f(x), \tag{4.8}$$

with the error again a factor $1+O(\epsilon \log \epsilon)$.

Now we can verify that the solution of (4.8) is the potential $\phi_b = \phi_{2D} + f(x)$. Substituting this assumed potential in (4.8) we have

$$\phi_{2D} + f(x) = -\frac{1}{2\pi}\int_{C+\bar{C}} \left(\frac{\partial \phi_{2D}}{\partial n} - \phi_{2D}\frac{\partial}{\partial n}\right) \log\left(\frac{\rho}{L}\right) dl + \frac{1}{2\pi} f(x) \int_{C+\bar{C}} \frac{\partial}{\partial n} \log\left(\frac{\rho}{L}\right) dl + f(x),$$

but ϕ_{2D} satisfies the classical integral equation

$$\phi_{2D} = -\frac{1}{2\pi}\int_{C+\bar{C}} \left(\frac{\partial \phi_{2D}}{\partial n} - \phi_{2D}\frac{\partial}{\partial n}\right) \log\frac{\rho}{L}\, dl, \tag{4.9}$$

† We use the term 'integral equation' here in a generalized sense, since the field point is not on the body. Clearly, however, equations of this sort are equivalent to integral equations since one may pass to the appropriate limit where the field point is on the body.

corresponding to two-dimensional flow past the double-body in an infinite fluid. Moreover,

$$\int_{C+\bar{C}} \frac{\partial}{\partial n} \log \frac{\rho}{L}\, dl = 0, \tag{4.10}$$

since this is the potential of a uniform normal dipole distribution over a closed contour, and the field point (y,z) is outside this contour. Thus we have verified that the potential

$$\phi_b = \phi_{2D} + f(x) \tag{4.11}$$

satisfies the integral equation (4.8) and is therefore the desired solution of our problem.

Repeating the above analysis with $F(x) = 0$, we obtain the classical slender body result

$$\phi_0 = \phi_{2D} + \frac{1}{2\pi} \int_{-\frac{1}{2}L}^{\frac{1}{2}L} \log \frac{2|x-\xi|}{L} \operatorname{sgn}(x-\xi) \frac{\partial}{\partial \xi} \left(\int_C \frac{\partial \phi_0}{\partial n}\, dl \right) d\xi. \tag{4.12}$$

It follows that the 'two-dimensional' solution (4.11) for the free-surface problem is equivalent, to the degree of approximation retained throughout, to the three-dimensional potential

$$\phi_b = \phi_0 + F(x), \tag{4.13}$$

and thus that (4.13) is also a solution of the original problem.

This completes the derivation of the results presented in § 3.

5. Bodies of revolution

While the solution presented in §§ 3 and 4 is valid for quite general slender bodies, the double-body potentials ϕ_0 and ϕ_{2D} cannot be obtained explicitly unless the transverse contours C are of simple geometrical shape (i.e. unless these can be mapped on the unit circle in closed form). Thus to proceed further for a general body, such as a ship hull, would require a numerical procedure. In order to continue the analysis we shall restrict ourselves to bodies of revolution. In this way the application of the results can be illustrated most easily, and comparison with Ursell's (1962) theory can be made.

The body will be defined by the equation

$$r = r_0(x), \quad \text{for} \quad -\tfrac{1}{2}L \leqslant x \leqslant \tfrac{1}{2}L, \tag{5.1}$$

where $r_0 = O(\epsilon)$. Here (r, θ) are polar co-ordinates about the x-axis, such that

$$y = r\sin\theta \quad -z = r\cos\theta.$$

We shall assume the boundary condition on the body to be of the form

$$\partial\phi_b/\partial r = V(x)\sin\theta + W(x)\cos\theta, \quad -\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi. \tag{5.2}$$

Thus $V(x)$ and $W(x)$ denote the transverse and vertical velocities at each section, respectively.

For the double-body potential ϕ_0 we require an even function of z, or a solution satisfying the boundary condition

$$\partial\phi_0/\partial r = V(x)\sin\theta + W(x)|\cos\theta|, \quad \text{for} \quad r = r_0(x) \quad \text{and} \quad 0 \leqslant \theta \leqslant 2\pi. \tag{5.3}$$

The required potential is well known (cf. Ward 1955, §9.3), the general form for a solution symmetric about $z = 0$ being

$$\phi_0(x,r,\theta) = 2a_0(x)\log\frac{L}{r} + 2\sum_1^\infty\left[\frac{a_n(x)\cos 2n\theta}{2nr^{2n}} + \frac{b_n(x)\sin(2n-1)\theta}{(2n-1)r^{2n-1}}\right] + \int_{-\infty}^{\infty}\frac{da_0(\xi)}{d\xi}\log\frac{L}{2|\xi-x|}\operatorname{sgn}(\xi-x)\,d\xi. \quad (5.4)$$

Differentiating (5.4) with respect to r and using the boundary condition (5.3) to evaluate the unknown coefficients, it follows that

$$a_0(x) = -\pi^{-1}r_0(x)\,W(x), \quad (5.5)$$

$$a_n(x) = \frac{2}{\pi}\frac{(-1)^n}{4n^2-1}[r_0(x)]^{2n+1}W(x), \quad (5.6)$$

$$b_1(x) = -\tfrac{1}{2}[r_0(x)]^2\,V(x), \quad (5.7)$$

and

$$b_n(x) = 0 \quad (n \neq 1). \quad (5.8)$$

Thus the potential ϕ_0 is given by

$$\phi_0 = -\frac{2}{\pi}W(x)\,r_0\log\frac{L}{r} + \frac{2}{\pi}W(x)\,r_0\sum_1^\infty\frac{(-1)^n}{n(4n^2-1)}\left(\frac{r_0}{r}\right)^{2n}\cos 2n\theta - \frac{r_0^2}{r}V(x)\sin\theta - \frac{1}{\pi}\int_{-\frac{1}{2}L}^{\frac{1}{2}L}\frac{d}{d\xi}[W(\xi)r_0(\xi)]\log\frac{L}{2|\xi-x|}\operatorname{sgn}(\xi-x)\,d\xi. \quad (5.9)$$

To this we must add the free surface contribution

$$F(x) = \tfrac{1}{4}K\int_{-\frac{1}{2}L}^{\frac{1}{2}L}\left(\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}r_0(\xi)\frac{\partial\phi_b}{\partial n}d\theta\right)\{-\mathbf{H}_0(K|x-\xi|) - Y_0(K|x-\xi|) + 2iJ_0(K|x-\xi|)\}\,d\xi$$

$$= -\tfrac{1}{2}K\int_{-\frac{1}{2}L}^{\frac{1}{2}L}W(\xi)r_0(\xi)\{-\mathbf{H}_0 - Y_0 + 2iJ_0\}\,d\xi. \quad (5.10)$$

Thus for a body of revolution the complete first-order near-field body potential is

$$\phi_b = -\frac{2}{\pi}W(x)r_0(x)\log\frac{L}{r} + \frac{2}{\pi}W(x)r_0(x)\sum_1^\infty\frac{(-1)^n}{n(4n^2-1)}\left(\frac{r_0}{r}\right)^{2n}\cos 2n\theta - \frac{1}{\pi}\int_{-\frac{1}{2}L}^{\frac{1}{2}L}\frac{d}{d\xi}[W(\xi)r_0(\xi)]\log\frac{L}{2|\xi-x|}\operatorname{sgn}(\xi-x)\,d\xi - \tfrac{1}{2}K\int_{-\frac{1}{2}L}^{\frac{1}{2}L}W(\xi)\,r_0(\xi)\{-\mathbf{H}_0 - Y_0 + 2iJ_0\}\,d\xi - \frac{r_0^2}{r}V(x)\sin\theta, \quad (5.11)$$

with an error of order $\epsilon^2\log^2\epsilon$.

Finally, we note that

$$\{-\mathbf{H}_0(K|x-\xi|) - Y_0(K|x-\xi|) + 2iJ_0(K|x-\xi|)\}$$
$$= 2iH_0^{(1)}(K|x-\xi|) - \mathbf{H}_0(K|x-\xi|) + Y_0(K|x-\xi|)$$
$$= 2iH_0^{(1)}(K|x-\xi|) - \frac{2}{\pi}\int_0^\infty\frac{e^{-tK|x-\xi|}}{(1+t^2)^{\frac{1}{2}}}\,dt$$
$$= 2iH_0^{(1)}(K|x-\xi|) + \frac{2}{\pi K}\operatorname{sgn}(x-\xi)\frac{d}{d\xi}s_3(K|x-\xi|),$$

where $H_0^{(1)}$ is the Hankel function, $H_0^{(1)} = J_0 + iY_0$, and

$$s_3(z) = \int_0^\infty \frac{1-e^{-tz}}{t(1+t^2)^{\frac{1}{2}}}\,dt.$$

Using these relations and integrating once by parts we can verify that, with $V(x) = 0$, equation (5.11) is equivalent to the first-order terms in Ursell's corresponding potential† (Ursell 1962, equation (3.19)). Thus we have verified that, to the degree of approximation retained here, our results are consistent with those of Ursell.

6. The forces and moments on a body of revolution

From the linearized Bernoulli equation, neglecting second-order terms in the oscillation amplitudes, the hydrodynamic pressure in a fluid of density ρ is given by

$$p = i\omega\rho\phi,$$

where we have suppressed the factor $e^{-i\omega t}$. Thus the four hydrodynamic forces and moments acting on the body are

$$\begin{pmatrix} F_y \\ M_z \end{pmatrix} = -i\omega\rho \int_{-\frac{1}{2}L}^{\frac{1}{2}L} \begin{pmatrix} 1 \\ x \end{pmatrix} r_0(x) \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \phi(x, r_0, \theta) \sin\theta \, d\theta \, dx, \tag{6.1}$$

$$\begin{pmatrix} F_z \\ M_y \end{pmatrix} = i\omega\rho \int_{-\frac{1}{2}L}^{\frac{1}{2}L} \begin{pmatrix} 1 \\ -x \end{pmatrix} r_0(x) \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \phi(x, r_0, \theta) \cos\theta \, d\theta \, dx, \tag{6.2}$$

where we delete the force F_x which is of higher order. Substituting for ϕ from equations (2.2) and (5.11) we obtain

$$\begin{pmatrix} F_y \\ M_z \end{pmatrix} = \frac{\pi}{2} i\omega\rho \int_{-\frac{1}{2}L}^{\frac{1}{2}L} \begin{pmatrix} 1 \\ x \end{pmatrix} r_0^2(x) \left[V(x) + A \sin\beta \, e^{iKx\cos\beta} \right] dx, \tag{6.3}$$

$$\begin{aligned}
\begin{pmatrix} F_z \\ M_y \end{pmatrix} = & -\frac{4}{\pi} i\omega\rho \int_{-\frac{1}{2}L}^{\frac{1}{2}L} \begin{pmatrix} 1 \\ -x \end{pmatrix} r_0^2(x) \left[W(x) \left(\log \frac{L}{4r_0} + \frac{3}{2} \right) + \frac{\pi^2}{8} \omega A \, e^{iKx\cos\beta} \right] dx \\
& - i\omega\rho \int_{-\frac{1}{2}L}^{\frac{1}{2}L} \begin{pmatrix} 1 \\ -x \end{pmatrix} r_0(x) \left\{ \frac{2}{\pi} \int_{-\frac{1}{2}L}^{\frac{1}{2}L} \frac{d}{d\xi} \left[r_0(\xi) \, W(\xi) \right] \log \frac{L}{2|\xi - x|} \operatorname{sgn}(\xi - x) \, d\xi \right. \\
& \left. + K \int_{-\frac{1}{2}L}^{\frac{1}{2}L} r_0(\xi) \, W(\xi) \left\{ -\mathbf{H}_0 - Y_0 + 2iJ_0 \right\} d\xi \right\} dx \\
& + 2i\rho g A \int_{-\frac{1}{2}L}^{\frac{1}{2}L} \begin{pmatrix} 1 \\ -x \end{pmatrix} r_0(x) \, e^{iKx\cos\beta} \, dx,
\end{aligned} \tag{6.4}$$

where we have used the relation

$$\sum_1^\infty \frac{1}{n(4n^2-1)^2} = \frac{3}{2} - 2\log 2.$$

† In fact the last term of (5.11) involving $V(x)$, or transverse oscillations, can be found from Ursell's analysis if all singularities therein are replaced by their transverse derivatives.

Now the practical case of a rigid body floating in waves can be considered. Let the amplitudes of the four oscillatory displacements of the body be

$$\eta = \eta_0 e^{-i\omega t} \quad \text{(sway)},$$
$$\zeta = \zeta_0 e^{-i\omega t} \quad \text{(heave)},$$
$$\chi = \chi_0 e^{-i\omega t} \quad \text{(pitch)},$$
$$\psi = \psi_0 e^{-i\omega t} \quad \text{(yaw)}.$$

Then the boundary condition on the body is

$$\begin{aligned} &i\omega(\zeta_0 - x\chi_0)\cos\theta - i\omega(\eta_0 + x\psi_0)\sin\theta \\ &\quad = \frac{\partial}{\partial r}\left\{\phi_b + \frac{gA}{\omega}\exp\left[K(-r\cos\theta + ix\cos\beta + ir\sin\theta\sin\beta)\right]\right\} \\ &\quad \cong \frac{\partial\phi_b}{\partial r} - \omega A(\cos\theta - i\sin\theta\sin\beta)\, e^{iKx\cos\beta}. \end{aligned} \tag{6.5}$$

Comparing (6.5) with (5.2) we see that the appropriate functions V and W are given by

$$V(x) = -i\omega(\eta_0 + x\psi_0 + A\sin\beta\, e^{iKx\cos\beta}), \tag{6.6}$$

$$W(x) = i\omega(\zeta_0 - x\chi_0 - iA\, e^{iKx\cos\beta}). \tag{6.7}$$

Substitution of (6.6) and (6.7) in (6.3) and (6.4) gives the total hydrodynamic forces acting on the body. After some reduction we obtain

$$\begin{pmatrix} F_y \\ M_z \end{pmatrix} = \frac{\pi}{2}\omega^2\rho\int_{-\frac{1}{2}L}^{\frac{1}{2}L}\begin{pmatrix} 1 \\ x \end{pmatrix} r_0^2(x)\left[\eta_0 + x\psi_0 + 2A\sin\beta\, e^{iKx\cos\beta}\right]dx, \tag{6.8}$$

$$\begin{aligned} \begin{pmatrix} F_z \\ M_y \end{pmatrix} = \frac{4}{\pi}\omega^2\rho\int_{-\frac{1}{2}L}^{\frac{1}{2}L}\begin{pmatrix} 1 \\ -x \end{pmatrix} r_0^2\Big[(\zeta_0 - x\chi_0)\left(\log\frac{L}{4r_0} + \frac{3}{2}\right) \\ - iA\, e^{iKx\cos\beta}\left(\log\frac{L}{4r_0} + \frac{3}{2} + \frac{\pi^2}{8}\right)\Big]dx + \omega^2\rho\int_{-\frac{1}{2}L}^{\frac{1}{2}L}\begin{pmatrix} 1 \\ -x \end{pmatrix} r_0(x) \\ \times\left\{\frac{2}{\pi}\int_{-\frac{1}{2}L}^{\frac{1}{2}L}\frac{d}{d\xi}\left[(\zeta_0 - \xi\chi_0 - iA\, e^{iKx\cos\beta})\, r_0(\xi)\right]\log\frac{L}{2|\xi - x|}\operatorname{sgn}(\xi - x)\, d\xi \right. \\ \left. + K\int_{-\frac{1}{2}L}^{\frac{1}{2}L}(\zeta_0 - \xi\chi_0 - iA\, e^{iK\xi\cos\beta})\, r_0(\xi)\left[-\mathbf{H}_0 - Y_0 + 2iJ_0\right]d\xi\right\}dx \\ + 2i\rho gA\int_{-\frac{1}{2}L}^{\frac{1}{2}L}\begin{pmatrix} 1 \\ -x \end{pmatrix} r_0(x)\, e^{iKx\cos\beta}\, dx. \end{aligned} \tag{6.9}$$

The total forces and moments acting on the body will consist of the hydrodynamic components, represented by (6.8) and (6.9), plus the inertial forces and moments due to the body's own mass and the hydrostatic restoring force and moment associated with the displacement of a floating body from its equilibrium position. The desired equations of motion are obtained by setting these total forces and moments equal to zero, signifying that there are no further restraints or external forces acting on the body. Since there is no coupling between the transverse (sway and yaw) and longitudinal (heave and pitch) modes, the four resulting equations of motion can be analysed and discussed as two separate pairs of equations.

First, we consider the transverse equations of motion, consisting of the hydrodynamic contributions from (6.8) plus the inertial force and moment associated with sway and yaw acceleration. (There is of course no hydrostatic force or moment in the transverse modes.) Since the inertial forces are of the same order as the body's mass, or $O(\epsilon^2)$, the resulting total forces are entirely of second order in ϵ, and a consistent pair of equations of motion follows. For a symmetrical body these will be uncoupled and we obtain the solutions

$$\eta_0 = -\frac{\pi\rho}{2M} A \sin\beta \int_{-\frac{1}{2}L}^{\frac{1}{2}L} r_0^2(x) \cos(Kx\cos\beta)\, dx, \qquad (6.10)$$

$$\psi_0 = -\frac{i\pi\rho}{I_{zzb}+I_{zzf}} A \sin\beta \int_{-\frac{1}{2}L}^{\frac{1}{2}L} x r_0^2(x) \sin(Kx\cos\beta)\, dx, \qquad (6.11)$$

where M = body mass,

I_{zzb} = moment of inertia of body,

I_{zzf} = moment of inertia of displaced fluid.

The equations of motion for pitch and heave are obtained by summing the hydrodynamic components (6.9), the corresponding inertial force and moment, and the hydrostatic restoring force and moment. These equations will *not* be homogeneous in ϵ since both the hydrostatic components and the exciting force and moment from the undisturbed incident wave (represented by the last term in (6.9)) are of first order in ϵ, while the remaining forces and moments are $O(\epsilon^2)$. Thus for vertical oscillations the hydrostatic restoring force and moment plus the exciting force and moment from the undisturbed incident wave pressure field will dominate the dynamic behaviour of the body and determine, in a relatively trivial manner, the pitch and heave motions of the body. All hydrodynamic effects due to the body's disturbance of the water and all inertial effects will be of higher order in the slenderness parameter ϵ.

7. Conclusions

Two essentially separate considerations should be discussed: viz. the development of the velocity potential due to the motions and wave diffraction of a floating slender body and, on the other hand, the application of this solution to the equations of motion for a slender body in waves.

With regard to the velocity potential, the first-order solution is presented in equation (3.5). This solution consists of the zero-frequency potential, or the potential of the double body, satisfying the boundary condition $\partial\phi/\partial z = 0$ on $z = 0$, plus a function $F(x)$ equal to an integral transform of the total velocity flux at each section of the body. For transverse motions, where the total flux vanishes, the function $F(x)$ is zero, and to leading order the near-field potential is identical to that of the double body in an infinite fluid, with no free-surface effects present. For motions in the longitudinal plane the function $F(x)$ is non-zero and represents the effect of the free surface, including an imaginary term which gives rise to a wave damping force and moment. For the special case of an axisymmetric body the potential reduces to that obtained in a somewhat different manner by Ursell (1962), who also included second-order effects. In comparison to Ursell's approach, the present method based upon Green's theorem has the advantage of being valid for non-axisymmetric bodies. In principle we could retain higher-order effects analogous to Ursell's, by an iterative process, but considerable algebraic effort would be required. The present method also permits the examination of end effects. In developing the velocity potential we have assumed that the body has pointed ends, but undoubtedly this restriction could be removed to permit blunt ends, by including additional end-effect terms such as are found in the Appendix. The final form of the forces and moments suggests that in fact these particular results are valid for a body with blunt ends, but this statement has not been proved.

With regard to the equations of motion for a floating slender body in waves, it is clear that our results are relatively trivial. For transverse oscillations, consistent results are obtained including the hydrodynamic forces, but these do not include any free surface effects, and in fact are identical to the classical 'strip-theory' approach. For longitudinal oscillations, free-surface effects including damping are present, but the total force is dominated by lower-order hydrostatic and wave-exciting forces, with inertial, added mass, and damping effects all of one order higher in the slenderness parameter. Thus to leading order there is no resonance frequency or phase shift.† Nor is it justifiable *a priori* to rectify this situation by including the derived second-order effects, for if these are significant in the equations of motion all higher-order terms must be retained, unless otherwise established as being unnecessary.‡ This situation is analogous to the basic objection raised to thin ship theory, where on the other hand inertial forces are included in the lowest-order equations, leading to a resonance frequency with undamped oscillations (cf. Newman 1961). Thus it is clear that, in its present form, slender-body theory has not overcome all the objections of thin-ship theory. The situation is similar in studies of the steady state wave resistance (Maruo 1962; Vossers 1962; Tuck 1963) where the leading order resistance from slender-body theory is simply the limiting value of Michell's integral, based upon thin-ship theory, as the draft tends to zero.

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† In fact there will be some phase variation for non-symmetrical bodies due to the frequency-dependent phase of the complex exciting force.

‡ A possible approach to this problem is that adopted by Vossers (1962), wherein $r_0/L \ll 1$ but $Kr_0 = O(1)$.

REFERENCES

GRIM, O. 1957 Durch Wellen an einem Schiffskörper erregte Kräfte. *Proc. Symp. on the Behaviour of Ships in a Seaway*, Wageningen, pp. 232–65.

GRIM, O. 1960 A method for a more precise computation of heaving and pitching motions both in smooth water and in waves. *Third Symposium on Naval Hydrodynamics*, Office of Naval Research.

MARUO, H. 1962 Calculation of the wave resistance of ships, the draft of which is as small as the beam. *J. Soc. nav. Archit. Japan*, **112**, 161–77.

NEWMAN, J. N. 1961 A linearized theory for the motion of a thin ship in regular waves. *J. Ship Res.* **5**, 34–55.

TUCK, E. O. 1963 The steady motion of a slender ship. Ph.D. Thesis, Cambridge University.

URSELL, F. 1962 Slender oscillating ships at zero forward speed. *J. Fluid Mech.* **14**, 496–516.

VOSSERS, G. 1962 Some applications of the slender body theory in ship hydrodynamics. Unpublished Thesis, Delft.

WARD, G. N. 1955 *Linearized Theory of Steady High-Speed Flow*, §9.3. Cambridge University Press.

WATSON, G. N. 1952 *Bessel Functions*, 2nd Ed. Cambridge University Press.

WEHAUSEN, J. V. & LAITONE, E. V. 1961 *Handbuch der Physik*, **9**, Surface Waves, §21. Berlin: Springer-Verlag.

Appendix

Here we shall derive the asymptotic expressions (4.1) and (4.2), for the potential of a distribution of sources or normal dipoles over the surface of a floating slender body. First, let us consider the case of a source distribution, of density $\sigma(S)$. Let the body surface S be defined by the (single-valued) function

$$r = r_0(\xi, \theta) \quad \text{for} \quad -\tfrac{1}{2}L \leqslant \xi \leqslant \tfrac{1}{2}L \quad \text{and} \quad -\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi.$$

Here (r, θ, ξ) are circular cylindrical co-ordinates, with

$$\eta = r\sin\theta \quad \text{and} \quad \zeta = -r\cos\theta.$$

The differential element dS of the surface S is

$$dS = r_0(\xi,\theta)\left[1+\left(\frac{\partial r_0}{\partial \xi}\right)^2+\frac{1}{r_0^2}\left(\frac{\partial r_0}{\partial \theta}\right)^2\right]^{\frac{1}{2}} d\theta\, d\xi.$$

With these substitutions the integral (3.5) is given by

$$I = \int_{-\frac{1}{2}L}^{\frac{1}{2}L} d\xi \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta\, \sigma(\xi,\theta)\,[(x-\xi)^2+\rho^2]^{-\frac{1}{2}} \\ r_0(\xi,\theta)\left[1+\left(\frac{\partial r_0}{\partial \xi}\right)^2+\frac{1}{r_0^2}\left(\frac{\partial r_0}{\partial \theta}\right)^2\right]^{\frac{1}{2}}, \quad \text{(A.1)}$$

where $\rho^2 = (y-\eta)^2+(z-\zeta)^2$. Making use of the relation

$$[(x-\xi)^2+\rho^2]^{-\frac{1}{2}} = -\operatorname{sgn}(x-\xi)\frac{\partial}{\partial \xi}\log\left[\frac{|x-\xi|+[(x-\xi)^2+\rho^2]^{\frac{1}{2}}}{L}\right],$$

and performing a partial integration with respect to ξ, we have

$$I = \int_{-\frac{1}{2}L}^{\frac{1}{2}L} d\xi \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta \log\left[\frac{|x-\xi| + [(x-\xi)^2+\rho^2]^{\frac{1}{2}}}{L}\right] \operatorname{sgn}(x-\xi)$$

$$\times \frac{\partial}{\partial\xi}\left\{\sigma(\xi,\theta)\, r_0(\xi,\theta)\left[1+\left(\frac{\partial r_0}{\partial\xi}\right)^2+\frac{1}{r_0^2}\left(\frac{\partial r_0}{\partial\theta}\right)^2\right]^{\frac{1}{2}}\right\}$$

$$-2\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta\, \sigma(x,\theta)\, r_0(x,\theta)\left[1+\left(\frac{\partial r_0}{\partial\xi}\right)^2+\frac{1}{r_0^2}\left(\frac{\partial r_0}{\partial\theta}\right)^2\right]^{\frac{1}{2}}_{\xi=x} \log\left(\frac{\rho}{L}\right)$$

$$+\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta\, \sigma(\tfrac{1}{2}L,\theta)\, r_0(\tfrac{1}{2}L,\theta)\left[1+\left(\frac{\partial r_0}{\partial\xi}\right)^2+\frac{1}{r_0^2}\left(\frac{\partial r_0}{\partial\theta}\right)^2\right]^{\frac{1}{2}}_{\xi=\frac{1}{2}L}$$

$$\times \log\left[\frac{\frac{1}{2}L-x+[(\frac{1}{2}L-x)^2+\rho^2]^{\frac{1}{2}}}{L}\right]$$

$$+\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta\, \sigma(-\tfrac{1}{2}L,\theta)\, r_0(-\tfrac{1}{2}L,\theta)\left[1+\left(\frac{\partial r_0}{\partial\xi}\right)^2+\frac{1}{r_0^2}\left(\frac{\partial r_0}{\partial\theta}\right)^2\right]^{\frac{1}{2}}_{\xi=-\frac{1}{2}L}$$

$$\times \log\left[\frac{\frac{1}{2}L+x+[(\frac{1}{2}L+x)^2+\rho^2]^{\frac{1}{2}}}{L}\right]. \qquad (A.2)$$

If we assume that the body is slender, with pointed ends, and that the field point is in the near field, it follows that

$$\rho = O(\epsilon), \quad \frac{\partial r_0}{\partial\xi} = O(\epsilon), \quad \text{and} \quad r_0(\pm\tfrac{1}{2}L) = 0.$$

Thus it follows that

$$I \cong \int_{-\frac{1}{2}L}^{\frac{1}{2}L} d\xi \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta \log\left[\frac{2|x-\xi|}{L}\right] \operatorname{sgn}(x-\xi)\frac{\partial}{\partial\xi}\,\sigma(\xi,\theta)\, r_0(\xi,\theta)\left[1+\frac{1}{r_0^2}\left(\frac{\partial r_0}{\partial\theta}\right)^2\right]^{\frac{1}{2}}$$

$$-2\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta\, \sigma(x,\theta)\, r_0(x,\theta)\left[1+\frac{1}{r_0^2}\left(\frac{\partial r_0}{\partial\theta}\right)^2\right]^{\frac{1}{2}}_{\xi=x} \log\left(\frac{\rho}{L}\right), \qquad (A.3)$$

with the error a factor $1+O(\epsilon^2)$. Since

$$r_0\left[1+\frac{1}{r_0^2}\left(\frac{\partial r_0}{\partial\theta}\right)^2\right]^{\frac{1}{2}} = dl,$$

the differential element of arc length along the contour C, the final result is obtained that

$$I \cong \int_{-\frac{1}{2}L}^{\frac{1}{2}L} d\xi \log\left[\frac{2|x-\xi|}{L}\right] \operatorname{sgn}(x-\xi)\frac{\partial}{\partial\xi}\left(\int_C \sigma(\xi,l)\,dl\right)$$

$$-2\int_C \sigma(x,l)\log(\rho/L)\,dl. \qquad (A.4)$$

Here the contour C is the intersection of the transverse plane x = const. with the submerged surface S. Thus for a floating body the contour C is open, but by a similar proof the same relation applies for a slender body in an infinite fluid (or totally submerged beneath the surface), in which case the contour C would be closed.

Next let us examine the corresponding asymptotic approximation for a normal dipole distribution. With the same notation we have

$$I = \iint_S \mu(S)\frac{\partial}{\partial n}[(x-\xi)^2+\rho^2]^{-\frac{1}{2}}\,dS$$
$$= \int_{-\frac{1}{2}L}^{\frac{1}{2}L} d\xi \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta\,\mu(\xi,\theta)\,r_0(\xi,\theta)\left(\frac{\partial}{\partial r}-\frac{1}{r_0}\frac{\partial r_0}{\partial \theta}\frac{\partial}{\partial \theta}-\frac{\partial r_0}{\partial x}\frac{\partial}{\partial x}\right)[(x-\xi)^2+\rho^2]^{-\frac{1}{2}}$$
$$= \int_{-\frac{1}{2}L}^{\frac{1}{2}L} d\xi \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta\,\mu(\xi,\theta)\,r_0(\xi,\theta)\left\{\left[1+\frac{1}{r_0^2}\left(\frac{\partial r_0}{\partial \theta}\right)^2\right]^{\frac{1}{2}}\frac{\partial}{\partial n_{2D}} - \frac{\partial r_0}{\partial x}\frac{\partial}{\partial x}\right\}[(x-\xi)^2+\rho^2]^{-\frac{1}{2}}, \quad \text{(A. 5)}$$

where n_{2D} denotes the two-dimensional normal in the plane $\xi = \text{const.}$, or

$$\frac{\partial}{\partial n_{2D}} = \left[1+\frac{1}{r_0^2}\left(\frac{\partial r_0}{\partial \theta}\right)^2\right]^{-\frac{1}{2}}\left[\frac{\partial}{\partial r}-\frac{1}{r_0}\left(\frac{\partial r_0}{\partial \theta}\right)\frac{\partial}{\partial \theta}\right].$$

The term in (A. 5) involving $\partial/\partial x$ can be treated by partial integration and the use of the relevant equations for a source distribution, and for the slender body with $r_0 = O(\epsilon)$ and $\partial r_0/\partial x = O(\epsilon)$, the contribution from this term will be $O(\mu\epsilon^2 \log \epsilon)$. Thus

$$I = -\int_{-\frac{1}{2}L}^{\frac{1}{2}L} d\xi \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta\,\mu(\xi,\theta)\,r_0(\xi,\theta)\left[1+\frac{1}{r_0^2}\left(\frac{\partial r_0}{\partial \theta}\right)^2\right]^{\frac{1}{2}} \times \frac{\rho}{[(x-\xi)^2+\rho^2]^{\frac{3}{2}}}\frac{\partial \rho}{\partial n_{2D}} + O(\mu\epsilon^2\log\epsilon). \quad \text{(A. 6)}$$

Making use of the relation

$$\frac{\rho}{[(x-\xi)^2+\rho^2]^{\frac{3}{2}}} = \operatorname{sgn}(x-\xi)\frac{\partial}{\partial \xi}\frac{\rho}{\{|x-\xi|+[(x-\xi)^2+\rho^2]\}\,[(x-\xi)^2+\rho^2]^{\frac{1}{2}}}$$
$$= \operatorname{sgn}(x-\xi)\frac{\partial}{\partial \xi}\operatorname{sgn}(x-\xi)\frac{\partial}{\partial \xi}\frac{\rho}{|x-\xi|+[(x-\xi)^2+\rho^2]^{\frac{1}{2}}}$$

and integrating twice by parts, it follows that

$$I = -\int_{-\frac{1}{2}L}^{\frac{1}{2}L} d\xi \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta\,\frac{\rho}{\{|x-\xi|+[(x-\xi)^2+\rho^2]^{\frac{1}{2}}\}}\frac{\partial^2}{\partial \xi^2} \times \left\{\mu(\xi,\theta)\,r_0(\xi,\theta)\frac{\partial \rho}{\partial n_{2D}}\left[1+\frac{1}{r_0^2}\left(\frac{\partial r_0}{\partial \theta}\right)^2\right]^{\frac{1}{2}}\right\}$$
$$-2\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta\,\mu(x,\theta)\,r_0(x,\theta)\left\{\frac{1}{\rho}\frac{\partial \rho}{\partial n_{2D}}\left[1+\frac{1}{r_0^2}\left(\frac{\partial r_0}{\partial \theta}\right)^2\right]^{\frac{1}{2}}\right\}_{\xi=x}$$
$$+\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta\left[\frac{\rho}{|x-\xi|+[(x-\xi)^2+\rho^2]^{\frac{1}{2}}}\,\mu(\xi,\theta)\frac{\partial r_0}{\partial \xi}\frac{\partial \rho}{\partial r_{2D}}\left[1+\frac{1}{r_0^2}\left(\frac{\partial r_0}{\partial \theta}\right)^2\right]^{\frac{1}{2}}\right]_{\xi=-\frac{1}{2}L}^{\xi=\frac{1}{2}L}$$
$$+O(\mu\epsilon^2\log\epsilon), \quad \text{(A. 7)}$$

where we have assumed that $r_0(\pm\frac{1}{2}L) = 0$.

The second term in the last equation is equal to

$$-2\int_C \mu(x,l)\frac{\partial}{\partial n}\log(\rho/L)\,dl = O(\mu),$$

while the first term is $O(\mu\epsilon^2\log\epsilon)$. The last term is $O(\mu\epsilon^2)$ for $-\frac{1}{2}L < x < \frac{1}{2}L$ and $O(\mu\epsilon)$ for $x = \pm\frac{1}{2}L$. Therefore

$$I \cong -2\int_C \mu(x,l)\frac{\partial}{\partial n}\log(\rho/L)\,dl, \tag{A.8}$$

with the error a factor $1+O(\epsilon^2\log\epsilon)$ except near the ends, where for pointed bodies the error is a factor $1+O(\epsilon)$.